Line-Integral Formulation of the Hybrid MM/FEM Technique

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Abstract — **We apply a line-integral formulation to the Hybrid Mode-Matching/FEM Technique for the analysis of discontinuities with waveguides of arbitrary cross-section. The line-integral formulation is used instead of standard surface integrals, not only for the evaluation of coupling integrals but also for normalization integrals computation. In this way a noticeable advantage is gained in terms of numerical efficiency, particularly when accurate solutions are sought for. Numerical examples validate the proposed approach.**

I. Introduction

The Mode-Matching (MM) efficiency makes this technique widely used for the analysis of waveguide discontinuities. As shown in [1] a complete line-integral formulation of MM, instead of standard surface integrals, leads to an even more efficient implementation of this technique. On the other hand, MM is of limited flexibility, since typically it applies to waveguides of separable cross-section for which modal expansion of electromagnetic fields are analytically available. An Hybrid MM/FEM technique has been proposed in [2] for the analysis of discontinuities with waveguides of arbitrary cross section; this technique combines the computational efficiency of modal analysis with the versatility and flexibility of the FEM approach. Furthermore, the use of edge elements in the FEM region, as suggested in [3], allows to achieve accurate results.

In this paper we apply the line-integral formulation to the hybrid MM/FEM technique showing that this formulation presents a significant numerical advantage. We use the line-integral formulation not only for the coupling integrals but also for the normalization integrals. In addition, a great care is exerted in the choice of the edge elements to be employed in the FEM region.

The paper is organized in the following way: after a brief recall of the line-integral formulation of Mode-Matching technique (Sec. 2), the FEM 2D chosen formulation is presented (Sec. 3). The formulas to apply line-integral formulation to the hybrid MM/FEM technique are then given (Sec. 4). Two examples (Sec. 5) show the accuracy and the advantages of the proposed method, by comparison with the standard surface integrals approach. Finally, Sec. 6 reports the conclusions.

II. Line-Integral Formulation of MM Technique

The field components of a general cylindrical waveguide, can be expressed in terms of the Hertz-type potentials φ (for TE modes) and ψ (for TM modes), as follows:

$$\begin{cases} \underline{H_t} = \nabla_t \varphi \\ \underline{E_t} = Z_h \underline{a_z} \times \underline{H_t} \\ H_z = j\dfrac{k_c^2}{\beta}\varphi \end{cases} \qquad \text{(TE; z = 0)} \qquad (1)$$

$$\begin{cases} \underline{E_t} = \nabla_t \psi \\ \underline{H_t} = -Y_e \underline{a_z} \times \underline{E_t} \\ E_z = \dfrac{k_c^2}{\beta}\psi \end{cases} \qquad \text{(TM; z = 0)} \qquad (2)$$

where: $\underline{a}_z$ is the normal unit vector being z the propagation axis, Z_h the scalar wave impedance, β the propagation constant, k_c the cutoff wavenumber and Y_e the scalar wave admittance.

Referring to Fig. 1 and using (1) and (2), the following formulas for the coupling and the normalization integrals can be derived [1]:

Coupling integrals:

$$\frac{k_{c2}^2}{k_{c2}^2 - k_{c1}^2}\oint \varphi^{(2)} \frac{d\varphi^{(1)}}{dn} dc \qquad \text{(TE-TE Modes)} \qquad (3)$$

$$\frac{k_{c1}^2}{k_{c1}^2 - k_{c2}^2}\oint \psi^{(1)} \frac{d\psi^{(2)}}{dn} dc \qquad \text{(TM-TM Modes)} \qquad (4)$$

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$$\oint \psi^{(1)} \frac{d\varphi^{(2)}}{d\tau} dc \quad \text{(TM-TE Modes)} \qquad (5)$$

Normalization integrals:

$$\int_S H_t \cdot H_t dS = -\frac{k_c}{2} \oint_C \varphi \frac{\partial^2 \varphi}{\partial n \partial k_c} dc \quad \text{(TE Modes)} \qquad (6)$$

$$\int_S E_t \cdot E_t dS = \frac{k_c}{2} \oint_C \frac{\partial \psi}{\partial n} \frac{\partial \psi}{\partial k_c} dc \quad \text{(TM Modes)} \qquad (7)$$

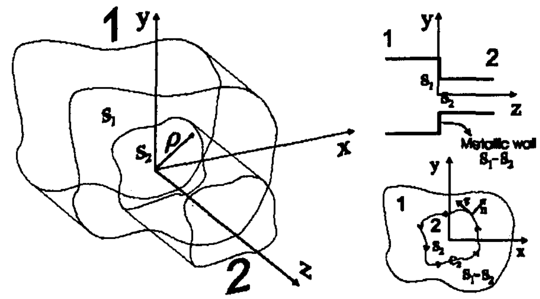

Fig. 1 A general waveguide discontinuity

III. FEM 2D FORMULATION

Different formulations for the Finite Element Analysis of waveguides have been proposed in the literature [4]. We have chosen to implement the two dimensional "Standard Formulation" [5] that uses the 3 components of the magnetic field. It is able to cope with situations where ε and μ both vary within the waveguide profile, it allows the analysis of waveguides having sharp reentrant corners and, thanks to the use of appropriate "edge elements", it doesn't present spurious modes.

The magnetic field is splitted into its transverse and longitudinal components, $\underline{H} = \underline{H}_t + H_z \underline{a}_z$, that are evaluated by means of the following expressions:

$$\underline{h}_t = \sum_{j=1}^{N_v^e} h_{tj} \underline{\tau}_j \qquad h_z = \sum_{j=1}^{N_n^e} h_{zj} \alpha_j \qquad (8)$$

where N_v^e vector interpolation functions τ (edge elements) and N_n^e nodal interpolation functions α are to be associated with an element, $\underline{h}_t = \underline{H}_t \beta$ and $-jH_z = h_z$.

Imposing the tangential continuity and using the Galerking weighted residual method the electromagnetic problem is transformed into the problem represented by the following system matrix equation:

$$\begin{cases} A\underline{h}_t - k_0^2 B\underline{h}_t + \beta^2 C\underline{h}_z + \beta^2 D\underline{h}_t = 0 \\ E\underline{h}_z - k_0^2 F\underline{h}_z + C^T \underline{h}_t = 0 \end{cases} \qquad (9)$$

that may be written as:

$$\begin{bmatrix} A + \beta^2 D & \beta^2 C \\ C^T & E \end{bmatrix} \begin{bmatrix} \underline{h}_t \\ \underline{h}_z \end{bmatrix} = k_0^2 \begin{bmatrix} B & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} \underline{h}_t \\ \underline{h}_z \end{bmatrix} \qquad (10)$$

where the matrices [A] to [F] are the global matrices achieved assembling element by element starting from the matrices related to each individual element Ω_e [5].

The solution of (10) gives the eigenvalues k_0 and the coefficients h_t and h_z of the magnetic field expansions (8). From the knowledge of k_0 the cut-off wavenumber k_c can be derived: in fact, the TE and TM modes in a lossless and air-filled waveguide satisfy the dispersion relation:

$$k_c^2 - (j\beta)^2 = k_0^2 \qquad (11)$$

A great care must be exerted in the choice of the vector and nodal interpolation functions, τ and α.

Concerning the edge elements, τ, they come in various forms [6,7]. A very good accuracy and efficiency can be achieved by using the so called Linear Tangential, Quadratic Normal (LT/QN) basis functions, that in simplex coordinate form may be written as follows [8]:

$$\begin{aligned} \tau_1 &= l_1 \lambda_2 \nabla \lambda_3 \\ \tau_2 &= l_1 \lambda_3 \nabla \lambda_2 \\ \tau_3 &= l_2 \lambda_1 \nabla \lambda_3 \\ \tau_4 &= l_2 \lambda_3 \nabla \lambda_1 \\ \tau_5 &= l_3 \lambda_1 \nabla \lambda_2 \\ \tau_6 &= l_3 \lambda_2 \nabla \lambda_1 \\ \tau_7 &= c_{71} \lambda_1 \lambda_2 \nabla \lambda_3 - c_{72} \lambda_1 \lambda_3 \nabla \lambda_2 \\ \tau_8 &= c_{81} \lambda_1 \lambda_2 \nabla \lambda_3 - c_{82} \lambda_2 \lambda_3 \nabla \lambda_1 \end{aligned} \qquad (12)$$

where l_i are the lengths of the edges, λ_i the simplex coordinates associated with the nodes and c_{71}, c_{72}, c_{81} and c_{82} are normalization constants.

Also for the nodal interpolation functions α, a good trade-off between accuracy and complexity is represented by 2[nd] order functions. Their expressions are:

$$\begin{aligned}\alpha_1 &= \lambda_1(2\lambda_1 - 1)\\ \alpha_2 &= 4\lambda_1\lambda_2\\ \alpha_3 &= 4\lambda_1\lambda_3\\ \alpha_4 &= \lambda_2(2\lambda_2 - 1)\\ \alpha_5 &= 4\lambda_2\lambda_3\\ \alpha_6 &= \lambda_3(2\lambda_3 - 1)\end{aligned} \tag{13}$$

N_v^e and N_n^e in the (8) are then equal to 8 and 6, respectively.

IV. Application of Line-Integral Formulation to MM/FEM Technique

In order to use the formulas from (3) to (8), the potentials φ (for TE modes) and ψ (for TM modes) have to be expressed in terms of the transverse or the longitudinal components of the magnetic field.

For the TE modes from (1) and (8) one derives:

$$\varphi = \frac{\beta}{k_c^2}\frac{h_z}{\sqrt{C}} = \frac{\beta}{\sqrt{C}k_c^2}\sum_{n=1}^{T}\sum_{j=1}^{N_n^e} h_{zj}\alpha_j \tag{14}$$

where T is the number of the edges of the triangles on the boundary and C the normalization constant.
Hence,

$$\frac{\partial\varphi}{\partial n} = \frac{\beta}{\sqrt{C}k_c^2}\sum_{n=1}^{T}\sum_{j=1}^{N_n^e} h_{zj}\frac{\partial\alpha_j}{\partial n} = \frac{\beta}{\sqrt{C}k_c^2}\sum_{n=1}^{T}\sum_{j=1}^{N_n^e} h_{zj}\cdot \left(\frac{\partial\alpha_j}{\partial\lambda_1}\frac{\partial\lambda_1}{\partial n} + \frac{\partial\alpha_j}{\partial\lambda_2}\frac{\partial\lambda_2}{\partial n} + \frac{\partial\alpha_j}{\partial\lambda_3}\frac{\partial\lambda_3}{\partial n}\right) \tag{15}$$

Similarly for $\dfrac{\partial\varphi}{\partial\tau}$.

For the TM modes from (2) and (8) one obtains:

$$\psi = \frac{\beta}{\sqrt{C}k_c^2}E_z = \frac{\beta}{\sqrt{C}k_c^2}\frac{1}{j\omega\varepsilon}\nabla\times H_t = $$

$$= \frac{1}{\sqrt{C}\beta k_c^2}\sum_{n=1}^{T}\sum_{j=1}^{N_v^e} h_{tj}\nabla\times\tau_j \tag{16}$$

Only $\nabla\times\tau_7$ and $\nabla\times\tau_8$ contribute to $\dfrac{\partial\psi}{\partial n}$.

$$\frac{\partial\nabla\times\tau_7}{\partial n} = c_{71}(b_3c_1 - c_3b_1)\frac{\partial\lambda_2}{\partial n} + c_{71}(b_2c_1 - c_2b_1)\frac{\partial\lambda_3}{\partial n} - $$
$$- c_{72}(b_2c_3 - c_2b_3)\frac{\partial\lambda_1}{\partial n} - c_{72}(b_1c_3 - c_1b_3)\frac{\partial\lambda_2}{\partial n} \tag{17}$$

$$\frac{\partial\nabla\times\tau_8}{\partial n} = c_{81}(b_3c_2 - c_3b_2)\frac{\partial\lambda_1}{\partial n} + c_{81}(b_1c_2 - c_1b_2)\frac{\partial\lambda_3}{\partial n} - $$
$$- c_{82}(b_2c_3 - c_2b_3)\frac{\partial\lambda_1}{\partial n} - c_{82}(b_1c_3 - c_1b_3)\frac{\partial\lambda_2}{\partial n}$$

where a_i, b_i and c_i (i=1..3) are the coefficients to convert the Cartesian coordinates to the simplex coordinates [5].

In order to use the formulas (6) and (7), we apply the following expression that gives the Hertz-type potentials of the TE and TM modes versus the kc

$$\varphi(k_c, x, y) = \varphi\left(\overline{k_c}, \frac{k_c}{\overline{k_c}}x, \frac{k_c}{\overline{k_c}}y\right) \tag{18}$$

where $\overline{k_c}$ is the cut-off wavenumber of the various modes of the waveguide.

The formula (18) derives from the consideration that the cut-off wavenumbers are inversely proportional to the section size; furthermore, the potential in any point of a section has the same value of the potential in the corresponding point of another section with the same shape but different size.

From (18) it can be obtained:

$$\frac{\partial}{\partial k_c}\varphi(k_c, x, y)\Big|_{k_c=\overline{k_c}} = \frac{1}{\overline{k_c}}\nabla\varphi(\overline{k_c}, x, y)\bullet(x, y) \tag{19}$$

Hence, for example, in the TE case we have:

$$\frac{1}{\overline{k_c}}\nabla\varphi(\overline{k_c}, x, y)\bullet(x, y) = \frac{\beta}{\overline{k_c}^3}\sum_n\sum_i h_z(i)\nabla\alpha_i\bullet(x, y)$$
$$= \frac{\beta}{\overline{k_c}^3}\sum_n\sum_i h_z(i)\cdot\left[\left(\frac{\partial\alpha_i}{\partial\lambda_1}\nabla\lambda_1\bullet(x, y) + \right.\right.$$
$$\left.\left. + \frac{\partial\alpha_i}{\partial\lambda_2}\nabla\lambda_2\bullet(x, y) + \frac{\partial\alpha_i}{\partial\lambda_3}\nabla\lambda_3\bullet(x, y)\right]\right) \tag{20}$$

V. Numerical Results

To validate the proposed approach, we have considered the examples of a WR75 rectangular waveguide connected both with an elliptical waveguide and with a double ridge waveguide. We have evaluated the S11 of these structures with the Hybrid MM/FEM technique with line-integrals formulation as well as with the surface integrals formulation. We have compared the results with those achieved by means of the commercial tool HFSS (by Ansoft) and, for the first structure, by means of a pure MM tool [9].

The S11 results are shown in Fig. 2 and Fig. 3 for the first and the second example, respectively. In the figures FEM 2D IL and FEM 2D IS mean Hybrid MM/FEM technique with line integrals formulation and Hybrid MM/FEM technique with surface integrals formulation, respectively.

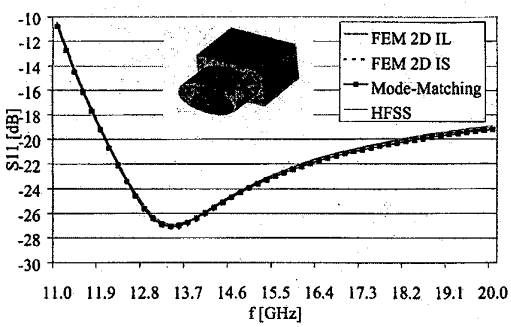

Fig. 2 S11 in the case of WR75/Elliptical discontinuity

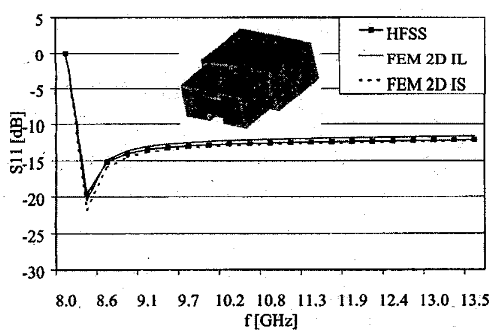

Fig. 3 S11 in the case of WR75/Double Ridge discontinuity

We have meshed the FEM region with 1073 triangles in the elliptical waveguide case and with 2100 triangles in the double ridge example.

In both the examples we have considered 20 Modes in the rectangular waveguide region and 16 modes in the FEM region.

The simulation time required by the Hybrid MM/FEM technique with line integrals formulation was 50% less than the one required with the surface integrals formulation. As can be seen from Fig. 2 and Fig. 3, a very good accuracy has been achieved.

This accuracy together with the advantage in terms of simulation time demonstrates the effectiveness of the proposed approach.

VI. Conclusion

The application of the line-integral formulation to the hybrid MM/FEM technique has been proposed. As demonstrated in Sec. 5, this formulation, together with a careful choice of the edge elements within the FEM region, is advantageous for the analysis of discontinuities with waveguides of arbitrary cross section, leading to a very efficient MM/FEM technique implementation.

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